On the existence of self-similar spherically symmetric wave maps coupled to gravity

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Abstract

We present a detailed analytical study of spherically symmetric self-similar solutions in the $SU(2)$ sigma model coupled to gravity. Using a shooting argument we prove that there is a countable family of solutions which are analytic inside the past self-similarity horizon. In addition, we show that for sufficiently small values of the coupling constant these solutions possess a regular future self-similarity horizon and thus are examples of naked singularities. One of the solutions constructed here has been recently found as the critical solution at the threshold of black hole formation.

1 Introduction

In this paper we continue our investigations, started in [1] (referred to as I), of wave maps coupled to gravity, that is solutions of Einstein's equations with an $SU(2)$ sigma field as matter. We found numerically in I that for $\alpha < 1/2$ (α is the dimensionless coupling constant) the model admits a countable family of continuously self-similar (CSS) solutions, labeled by an integer nodal index $n = 0, 1, \dots,$ that are analytic inside the past light cone of the singularity. We also provided evidence that the nth CSS solution can be continued up to the future light cone of the singularity if $\alpha < \alpha_n$, where $\{\alpha_n\}$ is an increasing sequence bounded above by $1/2$. The purpose of this paper is to make the results of I into theorem–proof rigorous mathematics. This is accomplished by applying a shooting argument to the resulting dynamical system. We note that the case $\alpha = 0$ was previously analyzed in [2].

The physical importance of the CSS solutions considered here was discussed in I, in particular we conjectured that in a certain parameter range ($\alpha_0 < \alpha < \alpha_1$) the $n = 1$ solution is a critical solution at the threshold of black hole formation. This conjecture has been recently confirmed in numerical studies of the critical behaviour [3] and in the linear stability analysis [4]. As far as we know, this is the only case where the existence of a self-similar solution, which was numerically found as the critical solution in gravitational collapse, has been established rigorously.

2 Setup

For the reader's convenience we repeat from I the basic setting for the problem. Let $X : M \to N$ be a map from a spacetime (M, g_{ab}) into a Riemannian manifold (N, G_{AB}). Wave maps coupled to gravity are defined as extrema of the action

$$S = \int_M \left(\frac{R}{16\pi G} + L_{WM} \right) dv_g \tag{1}$$

with the Lagrangian density

$$L_{WM} = -\frac{f_\pi^2}{2} g^{ab} \partial_a X^A \partial_b X^B G_{AB}. \tag{2}$$

Here G is Newton's constant and f_π^2 is the wave map coupling constant. The product $\alpha = 4\pi G f_\pi^2$ is dimensionless. The field equations derived from (1) are the wave map equation

$$\Box_g X^A + \Gamma^A_{BC}(X) \partial_a X^B \partial_b X^C g^{ab} = 0, \tag{3}$$

where $\Gamma^A_{BC}(X)$ are the Christoffel symbols of the target metric G_{AB} and $\Box_g$ is the d'Alembertian associated with the metric g_{ab}, and the Einstein equations $R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$ with the stress-energy tensor

$$T_{ab} = f_\pi^2 \left(\partial_a X^A \partial_b X^B - \frac{1}{2} g_{ab} (g^{cd} \partial_c X^A \partial_d X^B) \right) G_{AB}. \tag{4}$$

As a target manifold we take the three-sphere S^3 with the standard metric in polar coordinates $X^A = (F, \Theta, \Phi)$

$$G_{AB} dX^A dX^B = dF^2 + \sin^2 F \, (d\Theta^2 + \sin^2\Theta \, d\Phi^2). \tag{5}$$

For the domain manifold we assume spherical symmetry and use Schwarzschild coordinates

$$g_{ab} dx^a dx^b = -e^{-2\delta} A \, dt^2 + A^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{6}$$

where δ and A are functions of (t, r). Next, we assume that the wave maps are corotational, that is

$$F = F(t, r), \quad \Theta = \theta, \quad \Phi = \phi. \tag{7}$$

Equation (3) reduces then to the single semilinear wave equation

$$\Box_g F - \frac{\sin(2F)}{r^2} = 0, \tag{8}$$

where

$$\Box_g = -e^{\delta} \partial_t (e^{\delta} A^{-1} \partial_t) + \frac{e^{\delta}}{r^2} \partial_r (r^2 e^{-\delta} A \, \partial_r), \tag{9}$$

and the Einstein equations become

$$
\begin{aligned}
\partial_t A &= -2\alpha\, r A(\partial_t F)(\partial_r F), &(10)\\
\partial_r \delta &= -\alpha\, r \left((\partial_r F)^2 + A^{-2} e^{2\delta} (\partial_t F)^2\right), &(11)\\
\partial_r A &= \frac{1-A}{r} - \alpha\, r \left(A(\partial_r F)^2 + A^{-1} e^{2\delta}(\partial_t F)^2 + 2\,\frac{\sin^2 F}{r^2}\right). &(12)
\end{aligned}
$$

These equations are invariant under dilations $(t, r) \to (\lambda t, \lambda r)$ so it is natural to look for continuously self-similar (CSS) solutions, that is solutions which are left invariant by the action of the homothetic Killing vector $K = t\partial_t + r\partial_r$. To study such solutions it is convenient to use similarity variables $\rho = r/(-t)$ and $\tau = -\ln(-t)$. Then $K = -\partial_\tau$, so CSS solutions do not depend on τ. Assuming this and using an auxiliary function $Z = e^\delta \rho/A$, we reduce equations (8-12) to the system of ordinary differential equations (where prime is $d/d\rho$)

$$
\begin{aligned}
F'' &+ \frac{2}{\rho} F' - \alpha(1+Z^2)\rho F'^3 - \frac{\sin(2F)}{A\rho^2(1-Z^2)} = 0, &(13)\\
A' &= -2\alpha\rho A F'^2, &(14)\\
\rho Z' &= Z(1+\alpha(1-Z^2)\rho^2 F'^2), &(15)\\
\rho A' &= 1 - A - \alpha\left(\rho^2 A(1+Z^2)F'^2 + 2\sin^2 F\right). &(16)
\end{aligned}
$$

The combination of (14) and (16) yields the constraint

$$
1 - A - 2\alpha \sin^2 F + \alpha A \rho^2 F'^2 (1 - Z^2) = 0. \qquad (17)
$$

This system of equations has a fixed singularity at the center $\rho = 0$ and moving singularities at points where $Z(\rho) = \pm 1$ and/or $A(\rho) = 0$. In terms of the similarity coordinate ρ, the metric (6) takes the form

$$
ds^2 = A^{-1}(1 - Z^{-2})\, \rho^2 dt^2 + 2A^{-1} t\rho\, dt d\rho + A^{-1} t^2 d\rho^2 + t^2\rho^2 (d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (18)
$$

hence the hypersurfaces $Z = \pm 1$ are null (provided that $A > 0$). The first ρ_1 where $Z(\rho_1) = 1$ is the locus of the past light cone of the singularity at the origin $(t = 0, r = 0)$ (in what follows we shall refer to the past and future light cones of the singularity as to the past and future self-similarity horizons (SSH)). By rescaling, $\rho \to \rho/\rho_1$, one can always locate the past self-similarity horizon at $\rho_1 = 1$, that is $Z(1) = 1$. To ensure regularity of solutions in the interval $0 \le \rho \le 1$, the equations (13-17) must be supplemented by the boundary conditions at both endpoints

$$
\begin{aligned}
F(0) &= 0, \quad F'(0) = c, \quad Z(0) = 0, \quad A(0) = 1, &(19)\\
F(1) &= \frac{\pi}{2}, \quad F'(1) = b, \quad Z(1) = 1, \quad A(1) = 1 - 2\alpha, &(20)
\end{aligned}
$$

where c and b are free parameters.

Our main result is the following theorem:

Theorem 1. *For any $0 \leq \alpha < 1/2$ and any nonegative integer n, the equations (13-17) have an analytic solution (F_n, A_n, Z_n) which satisfies the boundary conditions (19-20) and has precisely n oscillations of $F_n(\rho)$ around $\pi/2$.*

In the next section we shall prove this theorem using a shooting technique. The numerical evidence for Theorem 1 was given in I . The case $\alpha = 0$ was proved previously in [2] so hereafter we assume that $0 < \alpha < 1/2$.

3 Proof of Theorem 1

For convenience we rewrite equations (13-15) in terms of $H = F - \pi/2$:

$$\begin{aligned} H'' &+ \frac{2}{\rho}H' - \alpha(1+Z^2)\rho H'^3 + \frac{\sin(2H)}{A\rho^2(1-Z^2)} = 0, &(21)\\ A' &= -2\alpha\rho A H'^2, &(22)\\ \rho Z' &= Z(1+\alpha(1-Z^2)\rho^2 H'^2), &(23) \end{aligned}$$

The constraint becomes

$$1 - 2\alpha - A + 2\alpha\sin^2 H + \alpha A\rho^2 H'^2(1-Z^2) = 0. \tag{24}$$

The initial conditions at $\rho = 0$ are

$$H(0) = -\frac{\pi}{2}, \quad H'(0) = c, \quad A(0) = 1, \quad Z(0) = 0, \quad Z'(0) = 1. \tag{25}$$

Note that the above equations have a residual scaling symmetry $\rho \to \lambda\rho$. The initial condition $Z'(0) = 1$ is imposed temporarily in order to fix the scale. We shall refer to solutions of equations (21-24) satisfying the initial conditions (25) as to c-orbits. In the appendix we show that c-orbits exist locally and are analytic in ρ and c. Now we shall show that c-orbits can be extended up to a point ρ_1 at which $Z(\rho_1) = 1$.

Proposition 2. *For any $0 < \alpha < 1/2$ and $c > 0$ there is a $\rho_1(c) \in (\sqrt{1-2\alpha}, 1)$, such that the c-orbit is defined for all $\rho < \rho_1$ and $\lim_{\rho\to\rho_1} Z(\rho) = 1$. Furthermore, the following limits exist*

$$-\frac{\pi}{2} < \bar{H} \overset{def}{=} \lim_{\rho\to\rho_1} H(\rho) < \frac{\pi}{2}, \quad \bar{A} \overset{def}{=} \lim_{\rho\to\rho_1} A(\rho) = 1 - 2\alpha\cos^2\bar{H}, \quad \lim_{\rho\to\rho_1}(1-Z^2)H'^2 = 0.$$

Proof: Let the maximum domain of definition of the c-orbit be $0 \leq \rho < \rho_1$ and assume that $Z(\rho) < 1$ in this interval. Then, from constraint (24) we have $A \geq 1 - 2\alpha > 0$ and hence $\bar{A} = \lim_{\rho\to\rho_1} A(\rho) > 0$ ($\bar{A}$ exists since $A(\rho)$ is monotone decreasing). By (23) $Z' \geq 0$, hence $\bar{Z} = \lim_{\rho\to\rho_1} Z(\rho)$ exists. If $\bar{Z} < 1$, then from constraint (24) H'^2 is bounded so $\bar{H} = \lim_{\rho\to\rho_1} H(\rho)$ exists, which in turn implies, again by (24), that $\lim_{\rho\to\rho_1} H'$ exits. Thus, $H, H', A,$ and Z all have finite limits at ρ_1 and therefore the c-orbit may be continued beyond ρ_1 contradicting the maximality of ρ_1. We conclude that $\bar{Z} = 1$.

Now, we must show that $\bar{H} \in (-\pi/2, \pi/2)$ exists. Since $\bar{Z} = 1$, we may no longer conclude that H'^2 is bounded but from equation (22) we get $(\ln A)' = -2\alpha\rho H'^2$, so H'^2 is integrable near ρ_1 which implies that H' is absolutely integrable ($|H'| < 1 + H'^2$) and thus $\bar{H}$ exists. From constraint (24), $H(\rho) = \pm\pi/2$ for some $0 < \rho < \rho_1$ is not possible since $1 - A > 0$. Thus, $-\pi/2 < H(\rho) < \pi/2$ and so $-\pi/2 \leq \bar{H} \leq \pi/2$. In fact, for $\rho \geq \rho_1/2$ we have $1 - A \geq \sigma > 0$, so $2\alpha \cos^2 H \geq \sigma > 0$ (remember that we assume $\alpha > 0$), hence H is uniformly bounded away from $\pm\pi/2$, and thus $-\pi/2 < \bar{H} < \pi/2$.

To prove $\bar{A} = 1 - 2\alpha \cos^2 \bar{H}$, note that by (24) $d = \lim_{\rho\to\rho_1} H'^2(1 - Z^2)$ exits and is finite. Hence, by (23) $\lim_{\rho\to\rho_1} Z'$ exists and is finite so $1 - Z^2 = O(\rho - \rho_1)$ near ρ_1. If $d \neq 0$, then $H'(\rho) \sim d/(\rho_1 - \rho)$ would not be integrable near ρ_1, thus d must be zero. Inserting this into (24) we get $\bar{A} = 1 - 2\alpha \cos^2 \bar{H}$.

Next, $(Z/\rho)' > 0$ by (23) and $\lim_{\rho\to 0}(Z/\rho) = 1$ by L'Hôpital's rule, hence $Z \geq \rho$ for all $\rho > 0$, and thus $\rho_1 \leq 1$. Finally, from (22) and (23)

$$\left(\frac{AZ^2}{\rho^2}\right)' = -\frac{2Z^4 A\alpha H'^2}{\rho} < 0, \tag{26}$$

and since $\lim_{\rho\to 0}(AZ^2/\rho^2) = 1$, we have $(AZ^2/\rho^2) \leq 1$ and hence $\rho_1 > \sqrt{A} > \sqrt{1 - 2\alpha}$.

If $Z(\rho_2) = 1$ for some $\rho_2 < \rho_1$, we replace ρ_1 by ρ_2 in the above arguments.

Corollary 3. *The function $\rho_1(c)$ is continuous.*

Proof: Let $\tilde{c}$ be given and let $\epsilon > 0$. By Proposition 2, $\rho_1(\tilde{c})$ is defined. The function $Z(\rho)$ is monotone increasing for $\rho < \rho_1(\tilde{c})$, so $Z(\rho_1(\tilde{c}) - \epsilon, \tilde{c}) < 1$, hence for all c sufficiently close to $\tilde{c}$, $Z(\rho_1(\tilde{c}) - \epsilon, c) < 1$, and thus $\rho_1(c) > \rho_1(\tilde{c}) - \epsilon$. To show that $\rho_1(c) < \rho_1(\tilde{c}) + \epsilon$ for all c sufficiently close to $\tilde{c}$, we assume otherwise and get a contradiction. By the mean-value theorem $Z(\rho_1(\tilde{c}) + \epsilon, c) - Z(\rho, c) = Z'(\xi, c)(\rho_1(\tilde{c}) + \epsilon - \rho)$. By continuity $Z(\rho, c)$ is close to $Z(\rho, \tilde{c})$ and $Z(\rho, \tilde{c})$ is close to 1 if ρ is close to $\rho_1(\tilde{c})$, hence $Z(\rho, c)$ is arbitrarily close to 1. But $Z'(\rho, c) > Z(\rho, c)/\rho > 1$, so $Z(\rho_1(\tilde{c}) + \epsilon, c) > Z(\rho, c) + \epsilon > 1$, which is a contradiction. Thus, $\rho_1(c) < \rho_1(\tilde{c}) + \epsilon$.

Lemma 4. *$H'(\rho)$ is bounded near ρ_1 if and only if $\bar{H} = 0$.*

Proof: Suppose that $\bar{H} \neq 0$ and $H'(\rho)$ is bounded. Then, in (21) we have

$$H'' = \text{bounded terms} - \frac{\sin 2H}{A\rho^2(1 - Z^2)} \sim \frac{d}{\rho_1 - \rho}, \tag{27}$$

where $d \neq 0$. This contradicts that $H'(\rho)$ is bounded near ρ_1 and concludes the "only if" part of Lemma 4.

Suppose now that $H(\rho_1) = 0$ and $H'(\rho)$ is unbounded. Without loss of generality we consider the case that $H(\rho) < 0$ and $H'(\rho) > 0$ near ρ_1. Dividing equation (21) by H' and integrating from ρ to ρ_1 we obtain

$$\int_\rho^{\rho_1} \left(\frac{H''}{H'} + \frac{2}{\rho} - \alpha(1 + Z^2)\rho H'^2 + \frac{\sin(2H)}{H' A\rho^2(1 - Z^2)}\right) d\rho = 0. \tag{28}$$

The first integral is divergent because $\lim_{\rho\to\rho_1} \ln H' = \infty$. The second and the third terms are integrable (remember that H'^2 is integrable). Thus, to get a contradiction it suffices to show that the last term is integrable. We write this term as

$$\frac{\sin(2H)}{H'A\rho^2(1-Z^2)} = \frac{\sin(2H)}{HA\rho^2}\frac{H}{(1-Z^2)H'}. \tag{29}$$

The first factor is continuous and we now show that the second factor is also continuous. Applying L'Hôpital's rule we get

$$\lim_{\rho\to\rho_1} \frac{H}{(1-Z^2)H'} = \lim_{\rho\to\rho_1} \frac{H'}{-2ZZ'H'+(1-Z^2)H''} = \lim_{\rho\to\rho_1} \frac{1}{-2ZZ'+(1-Z^2)H''/H'}. \tag{30}$$

Next, using (21) we get

$$(1-Z^2)\frac{H''}{H'} = -\frac{2(1-Z^2)}{\rho} + \alpha\rho(1+Z^2)(1-Z^2)H'^2 - \frac{\sin(2H)}{A\rho^2 H'}. \tag{31}$$

In the limit $\rho \to \rho_1$, the first term on the rhs of (31) obviously goes to zero, the second does by Proposition 2, and the third does by the assumption that $H' \to \infty$. Thus, the limit (30) is finite and consequently so is (29). This contradicts (28) and thus concludes the proof of the "if" part of Lemma 4.

Corollary 5. *A c-orbit which has $\bar{H}(c)=0$ is analytic on the whole interval $0 \le \rho \le \rho_1$.*

Proof: The boundedness of $H'(\rho)$ implies by (21) that $H'' > -2H'/\rho$ is bounded below (remember that $H(\rho) < 0$ and $H'(\rho) > 0$ near ρ_1), hence $\lim_{\rho\to\rho_1} H'(\rho)$ exists. Having that, it is easy to show by applying L'Hôpital's rule to $\lim_{\rho\to\rho_1}(H^{(k)}(\rho_1) - H^{(k)}(\rho))/(\rho_1-\rho)$ for $k=0,1$ that the solution (H, A, Z) is C^2 near ρ_1. By a routine contraction mapping argument one can show that C^2 solutions are unique, hence a c-orbit must belong to the one-parameter family of analytic solutions from Proposition 14 (see the appendix).

Next, we describe the behaviour of c-orbits for small and large values of the shooting parameter c. We define a nodal number of a c-orbit $N(c)$ = number of zeros of the function $H(\rho)$ on the interval $0 \le \rho < \rho_1$. We first show that c-orbits with small c have no nodes.

Proposition 6. *If c is sufficiently small then $N(c) = 0$.*

Proof: For $c = 0$ we have $H(\rho) \equiv -\pi/2$ and $Z(\rho) = \rho$ so $\rho_1(c=0) = 1$. By continuity, for any $\epsilon > 0$ and sufficiently small c we can find ρ_0 such that $1-\epsilon < \rho_0 < \rho_1(c) < 1$ and $H(\rho_0) < -\pi/2 + \epsilon$. We know from the proof of Proposition 2 that $\lim_{\rho\to\rho_1} \sqrt{\rho_1-\rho}\, H' = 0$, hence

$$H(\rho_1) - H(\rho_0) = \int_{\rho_0}^{\rho_1} H'(\rho)d\rho < const \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\rho_1-\rho}} < const\sqrt{\epsilon}. \tag{32}$$

Thus, $H(\rho)$ stays arbitrarily close to $-\pi/2$ all the way up to ρ_1 if c is sufficiently small and therefore $N(c) = 0$. We remark that using a scaling argument one can derive the precise asymptotic behaviour of c-orbits for small c. We omit this argument since it is not needed for the proof.

We show next that c-orbits with large c have arbitrarily many nodes.

Proposition 7. *$N(c) \to \infty$ for $c \to \infty$.*

Proof: We rescale the variables, setting

$$x = c\rho, \quad \tilde{H}(x) = H(\rho), \quad \tilde{A}(x) = A(\rho), \quad \tilde{Z}(x) = cZ(\rho). \tag{33}$$

Then, equations (21-24) become

$$\tilde{H}'' + \frac{2}{x}\tilde{H}' - \alpha(1+\frac{\tilde{Z}^2}{c^2})xH'^3 + \frac{\sin(2\tilde{H})}{\tilde{A}x^2(1-\frac{\tilde{Z}^2}{c^2})} = 0, \tag{34}$$

$$\tilde{A}' = -2\alpha x\tilde{A}\tilde{H}'^2, \tag{35}$$

$$x\tilde{Z}' = \tilde{Z}(1+\alpha(1-\frac{\tilde{Z}^2}{c^2})x^2\tilde{H}'^2), \tag{36}$$

with the constraint

$$1 - 2\alpha - \tilde{A} + 2\alpha\sin^2\tilde{H} + \alpha\tilde{A}x^2\tilde{H}'^2(1-\frac{\tilde{Z}^2}{c^2}) = 0, \tag{37}$$

and the initial conditions at $x = 0$

$$\tilde{H}(0) = -\frac{\pi}{2}, \quad \tilde{H}'(0) = 1, \quad \tilde{A}(0) = 1, \quad \tilde{Z}(0) = 0, \quad \tilde{Z}'(0) = 1. \tag{38}$$

As $c \to \infty$, the solutions of equations (34)-(38) tend uniformly on compact intervals to solutions of the limiting equations

$$h'' + \frac{2}{x}h' - \alpha xh'^3 + \frac{\sin(2h)}{ax^2} = 0, \tag{39}$$

$$a' = -2\alpha xah'^2, \tag{40}$$

$$xz' = z(1+\alpha x^2h'^2), \tag{41}$$

with the constraint

$$1 - 2\alpha - a + 2\alpha\sin^2 h + \alpha ax^2h'^2 = 0, \tag{42}$$

and the same initial conditions at $x = 0$

$$h(0) = -\frac{\pi}{2}, \quad h'(0) = 1, \quad a(0) = 1, \quad z(0) = 0, \quad z'(0) = 1. \tag{43}$$

We observe first that the function $a(x)$ is monotone decreasing by (40) and bounded below, $a > 1 - 2\alpha$, by (42). Thus, no singularity can develop due to a going to zero. Also, by (42) no singularity can develop due to h' becoming unbounded. Thus, solutions exist for all $x > 0$ (assuming the existence of a solution for small x). In order to complete the proof it is sufficient to show that the function $h(x)$ has an infinite number of zeros for $x > 0$. Since $a < 1$, it follows from (42) that $-\pi/2 < h(x) < \pi/2$ for all $x > 0$. To show that $h(x)$

oscillates around zero we consider three cases:
(i) Assume that $\lim_{x\to\infty} h(x)$ does not exist. Then, there must be a sequence $\ldots x_k < y_k < x_{k+1} < y_{k+1} < \ldots$ such that h has a local minimum at x_k and a local maximum at y_k. By (39), $h'(x_k) = 0, h''(x_k) \geq 0$ imply that $\sin(2h(x_k)) \leq 0$, hence $h(x_k) \leq 0$. By a similar argument, $h(y_k) \geq 0$. Thus, $h(x)$ has a zero in each interval $x_k < x < y_k$.
(ii) Assume that a nonzero $\lim_{x\to\infty} h(x)$ exists. Then, from (42) $\lim_{x\to\infty} x^2 h'^2$ exists and, in fact, equals zero because $\lim_{x\to\infty} h(x)$ exists. This implies by (39) that $\lim_{x\to\infty} x^2 h''(x) = -\sin(2h(\infty))/A(\infty) \neq 0$, hence $\lim_{x\to\infty} x^2 h'^2(x) \neq 0$. Thus the case (ii) does no arise.
(iii) Assume that $\lim_{x\to\infty} h(x) = 0$. We define the rotation function $\theta(x)$ by

$$\tan\theta(x) = \frac{xh'(x)}{h(x)}, \qquad \theta(0) = 0. \tag{44}$$

Remark 1. The rotation function $\theta(x)$ is well defined because the case $h(x) = h'(x) = 0$ is impossible for solutions satisfying the initial conditions (43). To see this, assume that $h(x_0) = h'(x_0) = 0$ for some $x_0 > 0$. Then, by (42) $a(x_0) = 1 - 2\alpha$ and the unique solution with these initial conditions at x_0 is $h(x) = 0, h'(x) = 0, a(x) = 1 - 2\alpha$ for all x, contradicting the initial conditions (43).

We want to show that $\lim_{x\to\infty} \theta(x) = -\infty$. Using (39) we obtain

$$x\theta'(x) = -\sin^2\theta - \frac{\sin 2h}{2h}\frac{2\cos^2\theta}{a} - \frac{(1 - 2\alpha\cos^2 h)\sin\theta\cos\theta}{a}. \tag{45}$$

Under the assumption $\lim_{x\to\infty} h(x) = 0$, it follows from (42) that $\lim_{x\to\infty} a(x) = 1 - 2\alpha$, hence for sufficiently large x

$$\theta'(x) \approx -\frac{1}{x}\left(\sin^2\theta + \sin\theta\cos\theta + \frac{2\cos^2\theta}{1 - 2\alpha}\right) < -\frac{3}{4x}, \tag{46}$$

so $\lim_{x\to\infty} \theta(x) = -\infty$. Thus, given any integer k there exists an x_k such that $h(x)$ has at least k zeroes for $x < x_k$. By continuous dependence on initial conditions, we may choose $c > x_k/\sqrt{1 - 2\alpha}$ so that the c-solution has k zeroes also for $x < x_k$. In terms of the variable $\rho = x/c$ the c-solution has k zeroes for $\rho < \sqrt{1 - 2\alpha} < \rho_1(c)$. This completes the proof of Proposition 7.

Next, we need two lemmas which tell us how the number of nodes $N(c)$ changes under small variations of c.

Lemma 8. *If $\bar{H}(\tilde{c}) = 0$, then $N(c) = N(\tilde{c})$ or $N(c) = N(\tilde{c}) + 1$ for c sufficiently close to $\tilde{c}$.*

Proof: First note that if $H(\rho, \tilde{c})$ has a zero at some $\rho_0 < \rho_1(\tilde{c})$, then $H'(\rho_0, \tilde{c}) \neq 0$ (see Remark 1) so by continuity of $H(\rho, c)$ with respect to c, $H(\rho, c)$ also has a zero if c is sufficiently close to $\tilde{c}$. Thus $N(c) \geq N(\tilde{c})$ and it suffices to show that $N(c) \leq N(\tilde{c}) + 1$. Let $\tilde{a} < \rho_1(\tilde{c})$ be the last node of the $\tilde{c}$-orbit, that is $H(\tilde{a}, \tilde{c}) = 0$ and, for concreteness, $H(\rho, \tilde{c}) < 0$ for $\tilde{a} < \rho < \rho_1$. By continuity with respect to c, $H(\rho, c)$ will also have a zero

at a near $\tilde{a}$ if c is near $\tilde{c}$. In order to prove that $H(\rho,c)$ cannot have more than one zero in the interval $a<\rho<\rho_1(c)$, we now show that if $H(\rho,c)$ becomes positive for some $\rho>a$, then it would not have time to change the sign again before going singular. Assume for contradiction that there is a segment $a<\rho_N\le\rho\le\rho_D$ of the c-orbit in which the function $H(\rho)$ is monotone decreasing from a local maximum $H(\rho_N)>0$ to $H(\rho_D)=0$.
We define

$$W=\frac{1}{2}\rho^2 A H'^2(1-Z^2)+\sin^2 H. \tag{47}$$

From (24) $W=(A-1+2\alpha)/(2\alpha)$, hence by (22) $W'<0$. We have

$$\frac{H'^2}{W-\sin^2 H}=\frac{2}{\rho^2 A(1-Z^2)},\quad \text{so}\quad \frac{-H'}{\sqrt{W-\sin^2 H}}=\frac{\sqrt{2}}{\rho\sqrt{A(1-Z^2)}}. \tag{48}$$

Integrating the left-hand side from ρ_N to ρ_D, we get (using $H_N=H(\rho_N)$)

$$\int_{\rho_N}^{\rho_D}\frac{-H'd\rho}{\sqrt{W-\sin^2 H}}=\int_0^{H_N}\frac{dH}{\sqrt{W-\sin^2 H}}\ge\int_0^{H_N}\frac{dH}{\sqrt{\sin^2 H_N-\sin^2 H}}>\frac{\pi}{2}, \tag{49}$$

where the first inequality follows from $W(\rho)\le W(\rho_N)=\sin^2 H_N$ (since W' decreases) and the second inequality is a simple calculation using a substitution $\sin H=u\sin H_N$ (remember that $H_N<\pi/2$).

Next, we derive an upper bound for the integral of the right-hand side of (48). We have

$$\int_{\rho_N}^{\rho_D}\frac{d\rho}{\rho\sqrt{A(1-Z^2)}}\le\frac{1}{\rho_N\sqrt{1-2\alpha}}\int_{\rho_N}^{\rho_D}\frac{d\rho}{\sqrt{1-Z^2}}\le\frac{1}{\rho_N\sqrt{1-2\alpha}}\int_{\rho_N}^{\rho_D}\frac{d\rho}{\sqrt{1-Z}}. \tag{50}$$

We showed above that $Z'>1$, hence $1-Z\ge\rho_1-\rho$. Therefore

$$\int_{\rho_N}^{\rho_D}\frac{d\rho}{\sqrt{1-Z}}\le\int_{\rho_N}^{\rho_D}\frac{d\rho}{\sqrt{\rho_1-\rho}}=2(\sqrt{\rho_1-\rho_N}-\sqrt{\rho_1-\rho_D})<2\sqrt{\rho_1-\rho_N}. \tag{51}$$

By continuity of solutions with respect to c and by Corollary 3, ρ_N is arbitrarily close to $\rho_1(c)$ if c is sufficiently close to $\tilde{c}$, hence it follows from (51) that the integral of the right-hand side of (48) is arbitrarily small. This is in contradiction with (49), hence $H(\rho,c)$ cannot have a second additional zero, which completes the proof of Lemma 8.

Lemma 9. *If $\bar{H}(\tilde{c})\neq 0$, then $N(c)=N(\tilde{c})$ for c sufficiently close to $\tilde{c}$.*

Proof: Without loss of generality we assume that $\bar{H}(\tilde{c})<0$. As above let $\tilde{a}<\rho_1(\tilde{c})$ be the last node of the $\tilde{c}$-orbit, that is $H(\tilde{a},\tilde{c})=0$ and $H(\rho,\tilde{c})<0$ for $\tilde{a}<\rho\le\rho_1$. Let a be the corresponding zero of $H(\rho,c)$ for c near $\tilde{c}$. We want to show that $H(\rho,c)$ cannot have an extra zero for $\rho>a$. Suppose for contradiction that $H(b,c)=0$ for some $b>a$. Then, there must be a $\delta<b$ such that $H(\delta,c)=\bar{H}(\tilde{c})$. Let us integrate the identity

$$\frac{H'}{\sqrt{W-\sin^2 H}}=\frac{\sqrt{2}}{\rho\sqrt{A(1-Z^2)}} \tag{52}$$

from δ to b. For the left hand side we get

$$\int_\delta^b \frac{H' d\rho}{\sqrt{W - \sin^2 H}} = \int_0^{-\bar{H}} \frac{dH}{\sqrt{W - \sin^2 H}}. \tag{53}$$

From Proposition 2 we know that $\lim_{\rho\to\rho_1}(1-Z^2)H'^2 = 0$ so $W(\rho,\tilde{c}) < (1+\epsilon/2)\sin^2\bar{H}$ for ρ near ρ_1 and hence $W(\rho, c) < (1+\epsilon)\sin^2\bar{H}$ for c near $\tilde{c}$. Since W is decreasing, $W(\delta, c) < W(\rho, c) < (1+\epsilon)\sin^2\bar{H}$. Thus

$$\int_0^{-\bar{H}} \frac{dH}{\sqrt{W - \sin^2 H}} \geq \int_0^{-\bar{H}} \frac{dH}{\sqrt{(1+\epsilon)\sin^2\bar{H} - \sin^2 H}} \geq \arcsin\left(\frac{1}{\sqrt{1+\epsilon}}\right) > \frac{\pi}{2} \tag{54}$$

for sufficiently small ϵ, where the last but one inequality can be seen by substituting $\sin H = u \sin\bar{H}$ into the integral. By the same argument as in the proof of Lemma 8, the integral of the right hand side of (52) is $O(\sqrt{\rho_1 - \rho})$. By continuity of solutions with respect to c and by Corollary 3, δ is arbitrarily close to $\rho_1(c)$ if c is sufficiently close to $\tilde{c}$, hence the integral of the left hand side of equation (52) is arbitrarily small. This contradicts (54) and completes the proof of Lemma 9.

Now we are ready to make a shooting argument. We define a set

$$C_0 = \{c \mid N(c) = 0\} \tag{55}$$

and let $c_0 = \sup C_0$. The set C_0 is nonempty (by Proposition 6) and bounded above (by Proposition 7) so c_0 exists. We claim that the c_0-orbit has no nodes and satisfies the boundary condition $\bar{H}(c_0) = 0$. To see this, note that the c_0-orbit cannot have a node because then by Lemmas 8 and 9 all nearby c-orbits would have a node so there would be an interval around c_0 without any elements of C_0 in it, contradicting the assumption that c_0 is the least upper bound. Thus, $N(c_0) = 0$. Now, if $\bar{H}(c_0) < 0$, then by Lemma 9 all nearby c-orbits would have no nodes, so there would be an interval around c_0 consisting of elements of C_0, contradicting the assumption that c_0 is an upper bound of C_0. Thus $\bar{H}(c_0) = 0$.

Next, we define $C_1 = \{c > c_0 \mid N(c) = 1\}$. This set is nonempty by the previous step and Lemma 8 and bounded above by Proposition 7, hence $c_1 = \sup C_1$ exists. By the same argument as above, the c_1-orbit has exactly one node and satisfies $\bar{H}(c_1) = 0$. The construction of subsequent c_n-orbits proceeds by induction.

Conclusion of the proof of Theorem 1:

Returning to the original variable $F(\rho)$ and rescaling $\rho \to \rho/\rho_1(c_n)$ we get the solution of equations (13-17) which satisfies the boundary conditions (19) and (20) and has exactly n intersections with the line $F = \pi/2$. By Corollary 5 this solution is analytic in the whole interval $0 \leq \rho \leq 1$.

4 Beyond the past self-similarity horizon

In this section we consider the behaviour of the CSS solutions of Theorem 1 outside the past SSH, in particular we ask the question: do these solutions possess a regular future self-similarity horizon? Note that $\rho = \infty$ corresponds to the hypersurface $(t = 0, r > 0)$ so in order to analyze the global behaviour of solutions (for $t > 0$) we need to go "beyond $\rho = \infty$". To this end we define, after I, a new coordinate x by

$$\frac{d}{dx} = \rho Z \frac{d}{d\rho}, \qquad x(\rho = 1) = 0. \tag{56}$$

We also define an auxiliary function $w(x) = 1/Z(\rho)$. In these new variables, the past SSH where $w = 1$ is at $x = 0$, while the future SSH (if it exists) is at some $x_A > 0$ where $w(x_A) = -1$.

In terms of x and w, the equations (21)-(23) become autonomous (where now prime is d/dx)

$$H'' - 2\alpha w H'^3 + \frac{\sin(2H)}{A(w^2-1)} = 0, \tag{57}$$

$$A' = -2\alpha A w H'^2, \tag{58}$$

$$w' = -1 + \alpha(1-w^2)H'^2. \tag{59}$$

The constraint (24) becomes

$$1 - 2\alpha - A + 2\alpha \sin^2 H + \alpha A H'^2 (w^2 - 1) = 0. \tag{60}$$

From (20) the initial conditions at $x = 0$ are

$$H(x) \sim bx, \quad w(x) \sim 1 - x, \quad A(x) \sim 1 - 2\alpha - 2\alpha(1-2\alpha)b^2 x. \tag{61}$$

We know from Theorem 1 that for each $\alpha < 1/2$ there is an infinite sequence $\{b_n(\alpha)\}$ determining solutions which are regular inside the past SSH, that is for all $x \leq 0$ (note that $\rho = 0$ corresponds to $x = -\infty$). In I we showed that for $x > 0$ the solutions starting from the past SSH with the initial conditions (61) tend in finite "time" to $w = -1$ if b is small, or to $w = +1$ if b is large. After I we shall refer to these two kinds of behaviour as to type A and type B solutions, respectively. Now we want to show that the solutions of Theorem 1 are of type A (and therefore possess the future SSH) provided that α is sufficiently small. Unfortunately, the shooting argument gives us insufficient information about the parameters b_n so we cannot apply the above mentioned result of I to determine the character of solutions outside the past SSH. Instead, we shall make use of the obvious fact that for $\alpha = 0$ all solutions are of type A.

Lemma 10. *For sufficiently small α the c_n-orbits of Theorem 1 (rescaled so that $\rho_1(c) = 1$) have $|b_n|$ uniformly bounded above for all n.*

Proof: It was shown in [2] (see Lemma 4 in that reference) that for $\alpha = 0$ the solution to equations (57)-(61) for $x < 0$ must exit the strip $|H| \leq \pi/2$ if $|b|$ is too large, say $|b| > B$. By continuous dependence, the same is true for sufficiently small α. But from Proposition 2 the c-orbits must stay in the strip $|H| \leq \pi/2$ for all $x < 0$. Thus, $|b_n| \leq B$ for small α.

Lemma 11. *If a solution to equations (57)-(60) has $w(x_0) < 0$ and $A(x_0) > 2/3$ for some x_0, then there is $x_A > x_0$ such that $\lim_{x \to x_A} w(x) = -1$, i.e., the solution is of type A.*

Proof: By (58) A is increasing for $w < 0$. Thus, using equation (59) and the constraint (60) we get for $x > x_0$

$$w' = -1 + \alpha(1 - w^2)H'^2 = -1 + \frac{1 - A - 2\alpha \cos^2 H}{A} < -2 + \frac{1}{A} \leq -\frac{1}{2}, \qquad (62)$$

hence w must hit -1 for some finite $x_A > x_0$.

Proposition 12. *The $c_n(\alpha)$-orbits are of type A if α is sufficiently small.*

Proof: For $\alpha = 0$ and any b we have $w(x) = 1 - x$ and $A(x) \equiv 1$; in particular $A(3/2) = 1 > 2/3$ and $w(3/2) = -1/2 < 0$. By continuous dependence on initial conditions there exists a $\delta(b)$ such that if $\alpha < \delta(b)$ and $|b - b'| < \delta(b)$, then $A(3/2, b') > 2/3$ and $w(3/2, b') < 0$. This implies by Lemma 11 that the solutions corresponding to such values of α and b' are of type A. By a standard theorem of advanced calculus there is a $\delta' > 0$ (independent of b) such that the solutions with $\alpha < \delta'$ and $|b| \leq B$ are of type A. By Lemma 10 any c_n-orbit has $|b| \leq B$, so for $\alpha < \delta'$ the c_n-orbits are of type A.

By a similar argument as in the proof of Proposition 2 one can easily show that the type A solutions are generically only C^0 at the future SSH (for isolated values of α there are solutions that go smoothly through the future SSH). In I we showed that the leading order asymptotic behaviour at the future SSH is (using $y = x_A - x$)

$$w \sim -1 + y, \quad A \sim A_0 - 2\alpha A_0 C^2 y \ln^2(y), \quad H \sim H_0 - C y \ln(y), \qquad (63)$$

where $A_0 = 1 - 2\alpha \cos^2 H_0$, $C = \sin(2H_0)/2A_0$, and H_0 is a free parameter. Using this expansion one can check that the curvature is finite as $y \to 0$ which means that the type A solutions are examples of naked singularities.

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Appendix (local existence theorems)

In [5] (Proposition 1) Breitelohner, Forgács, and Maison have derived the following result concerning the behaviour of solutions of a system of ordinary differential equations near a singular point (see also [6] for a similar result):

Theorem (BFM). Consider a system of first order differential equations for $n+m$ functions $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_m)$

$$t\frac{du_i}{dt} = t^{\mu_i} f_i(t, u, v), \qquad t\frac{dv_i}{dt} = -\lambda_i v_i + t^{\nu_i} g_i(t, u, v), \tag{64}$$

where constants $\lambda_i > 0$ and integers $\mu_i, \nu_i \geq 1$ and let C be an open subset of R^n such that the functions f and g are analytic in the neighbourhood of $t = 0, u = c, v = 0$ for all $c \in C$. Then there exists an n-parameter family of solutions of the system (64) such that

$$u_i(t) = c_i + O(t^{\mu_i}), \qquad v_i(t) = O(t^{\nu_i}), \tag{65}$$

where $u_i(t)$ and $v_i(t)$ are defined for all $c \in C, |t| < t_0(c)$ and are analytic in t and c.

We shall use this theorem to prove existence of local solutions of equations (21)-(23) near the singular points $\rho = 0$ and $\rho = 1$.

Proposition 13. *The equations (21)-(23) admit a two-parameter family of local solutions near $\rho = 0$*

$$\begin{aligned} H(\rho) &= -\frac{\pi}{2} + c\rho + O(\rho^3), && (66)\\ A(\rho) &= 1 - \alpha c^2 \rho^2 + O(\rho^4), && (67)\\ Z(\rho) &= d\rho + O(\rho^3), && (68) \end{aligned}$$

which are analytic in c, d and ρ.

Proof: Using the variables

$$w_1 = \frac{H + \pi/2}{\rho}, \quad w_2 = H', \quad w_3 = \frac{1-A}{\rho^2}, \quad w_4 = \frac{Z}{\rho} \tag{69}$$

we rewrite the equations (21)-(23) as the first order system

$$\begin{aligned} &\rho w_1' = -w_1 + w_2, && \rho w_2' = 2w_1 - 2w_2 + \rho^2 h_1,\\ &\rho w_3' = -2w_3 + 2\alpha w_2^2 + \rho^2 h_2, && \rho w_4' = \rho^2 h_3, \end{aligned} \tag{70}$$

where the functions h_i are analytic near $\rho = 0$. Next, substituting

$$\begin{aligned} &w_1 = u_1 - v_1, && w_2 = u_1 + 2v_1,\\ &w_3 = v_2 + \alpha(u_1^2 - 2v_1^2 - 8u_1 v_1), && w_4 = u_2 \end{aligned} \tag{71}$$

we put (70) into the form (64)

$$\rho u_1' = \rho^2 f_1, \qquad \rho u_2' = \rho^2 f_2,$$
$$\rho v_1' = -3v_1 + \rho^2 g_1, \qquad \rho v_2' = -2v_2 + \rho^2 g_2, \tag{72}$$

where the functions f_i, g_i are analytic in an open neighbourhood of $\rho = 0, u_1 = c, u_2 = d, v_i = 0$ for any c and d. Thus, according to the BFM theorem there exists a two-parameter family of solutions such that

$$u_1 = c + O(\rho^2), \qquad u_2 = d + O(\rho^2), \tag{73}$$
$$v_1 = O(\rho^2), \qquad v_2 = O(\rho^2), \tag{74}$$

which is equivalent to (66)-(68).

Proposition 14. *The equations (21)-(23) admit a one-parameter family of local solutions near* $\rho = 1$

$$H(\rho) = b(\rho - 1) + O((\rho - 1)^2), \tag{75}$$
$$A(\rho) = 1 - 2\alpha - 2\alpha(1 - 2\alpha)b^2(\rho - 1) + O((\rho - 1)^2), \tag{76}$$
$$Z(\rho) = \rho + O((\rho - 1)^2) \tag{77}$$

which are analytic in b *and* ρ.

Proof: We define the variables

$$u = H', \qquad v_1 = \frac{H}{\rho - 1} - H', \tag{78}$$
$$v_2 = \frac{(1 - 2\alpha) - A}{\rho - 1} - 2\alpha(1 - 2\alpha)H'^2, \qquad v_3 = \frac{Z - 1}{\rho - 1} - 1. \tag{79}$$

Then, the equations (21)-(23) take the form (using $t = \rho - 1$)

$$tu' = tf, \qquad tv_i' = -v_i + tg_i, \tag{80}$$

where the functions f and g_i are analytic in an open neighourhood of $t = 0, u = b, v_i = 0$ for any $b > 0$. Thus, according to the BFM theorem there exists a one-parameter family of solutions such that

$$u(t) = b + O(t), \qquad v_i(t) = O(t), \tag{81}$$

which is equivalent to (75)-(77).

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